Large Properties at Small Cardinals

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Université Paris Diderot - Paris 7
PhD thesis

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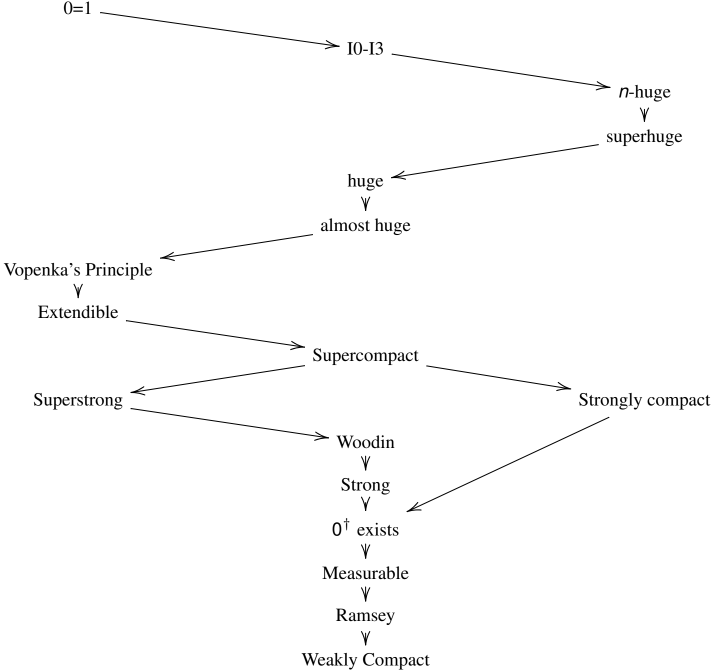

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n-huge
superhuge
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almost huge
Vopenka's Principle
Extendible
Supercompact
Superstrong
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Woodin
Strong
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Ramsey
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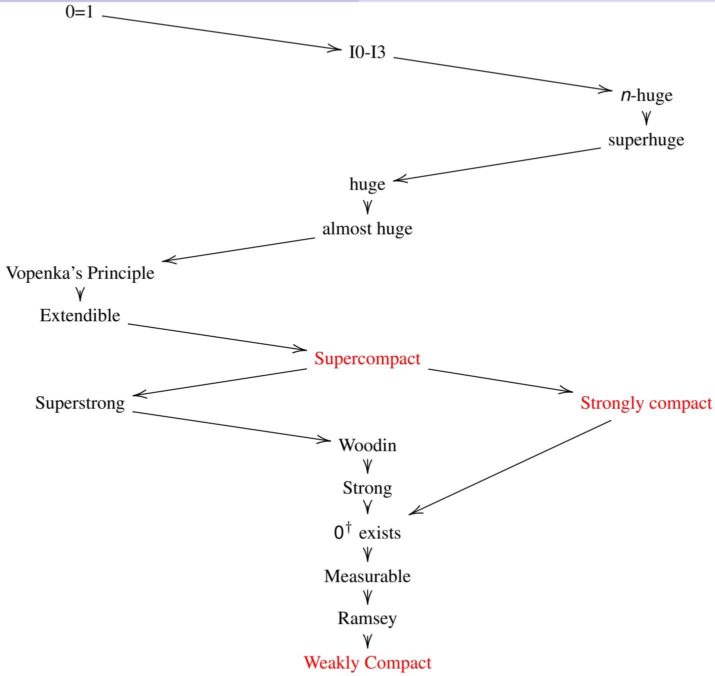
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κ is weakly compact iff it is inaccessible and it satisfies the tree property.

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Fontanella 2012 - Main Theorem 1

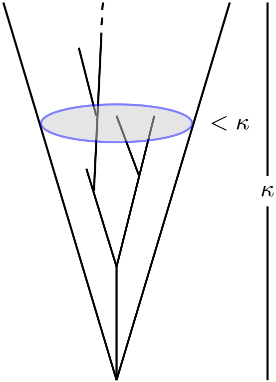
If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals$)$, then $Cons(ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property $)$.

Fontanella 2012 - Main Theorem 2

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals$)$, then $Cons(ZFC + \aleph_{\omega+1}$ has the Strong Tree Property $)$.

The Tree Property

A κ-tree, for a regular κ, is a tree of height κ and levels of size $< \kappa$.



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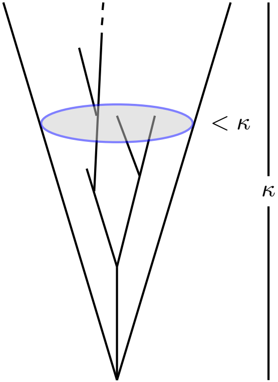
A regular cardinal κ satisfies the tree property if, and only if, every κ-tree has a cofinal branch.



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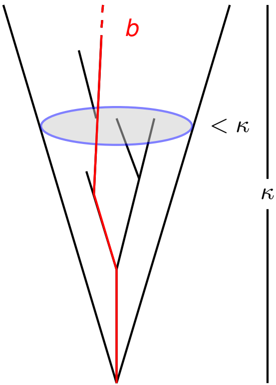

The Tree Property

Let κ be a regular cardinal.

Theorem

- (König's Lemma 1936) $\aleph_0$ satisfies the tree property;
- (Aronszajn 1934) $\aleph_1$ does not satisfy the tree property;
- (Specker 1949) If $\tau^{<\tau} = \tau$, then the tree property fails at τ^+;
- (Mitchell 1972) If *Cons*(*ZFC* $+ \exists\kappa$ weakly compact), then for every regular τ such that $\tau^{<\tau} = \tau$, we have *Cons*(*ZFC* $+ \tau^{++}$ has the tree property).

The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a (κ, λ)-tree is a subset $F \subseteq \{f : X \to 2;\ X \in [\lambda]^{<\kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq dom(f)$, then $f \restriction X \in F$;
2. for all $X \in [\lambda]^{<\kappa}$, $Lev_X(F) := \{f \in F;\ dom(f) = X\} \neq \emptyset$ and has size $< \kappa$.

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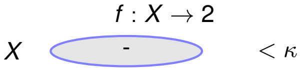

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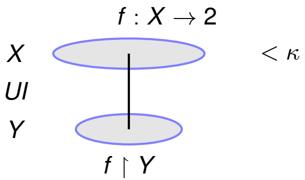

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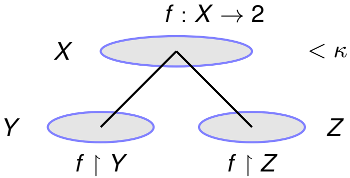

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Definition

A cofinal branch for a (κ, λ)-tree F is a function $b : \lambda \to 2$ such that $b \upharpoonright X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$

Definition

κ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every (κ, λ)-tree has a cofinal branch.

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The Super Tree Property

Definition

Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an F-level sequence if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

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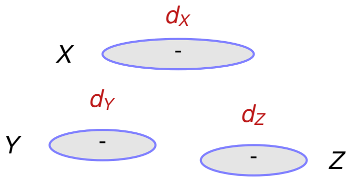
Let F be a (κ, λ)-tree. A sequence $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ is an *F-level sequence* if $d_X \in Lev_X(F)$, for all $X \in [\lambda]^{<\kappa}$.



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Definition

Let F be a (κ, λ)-tree and $D := \langle d_X;\ X \in [\lambda]^{<\kappa} \rangle$ an F-level sequence. An ineffable branch for D is a cofinal branch $b : \lambda \to 2$ such that

$$\{X \in [\lambda]^{<\kappa};\ b \upharpoonright X = d_X\}$$

is stationary.

Definition

κ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all (κ, λ)-tree F, every F-level sequence has an ineffable branch.

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The Tree Property at *Small Cardinals*

Mitchell 1972

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ weakly compact})$, then $Cons(ZFC + \aleph_n \text{ has the Tree Property })$.

Abraham 1983

If $Cons(ZFC + \exists\kappa < \lambda \text{ such that } \kappa \text{ is supercompact and } \lambda \text{ is weakly compact})$, then $Cons(ZFC + \aleph_2 \text{ and } \aleph_3 \text{ have the Tree Property })$.

Cummings & Foreman 1998

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \forall n \geq 2\ (\aleph_n \text{ has the tree property}))$.

The Tree Property at Small Cardinals

Magidor & Shelah 1996, Sinapova 2012

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then
$Cons(ZFC + \aleph_{\omega+1}$ has the tree property).

Neeman 2012

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then
$Cons(ZFC +$ every regular cardinal $\leq \aleph_{\omega+1}$ has the tree property).

Friedman & Halilović 2011

If $Cons(ZFC + \exists\langle\kappa$ weakly compact hypermeasurable), then
$Cons(ZFC + \aleph_{\omega+2}$ has the tree property).

Open Question

Is it possible to construct a model where all regular cardinals above $\aleph_1$ simultaneously satisfy the tree property?

Weiss 2010

Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

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Strong Tree Properties at Small Cardinals

Fontanella 2011

If $Cons(ZFC + \exists\kappa, \lambda \text{ supercompact cardinals })$, then $Cons(ZFC + \aleph_2 \text{ and } \aleph_3 \text{ have the Super Tree Property })$.

Fontanella 2012 - Main Theorem 1

If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \forall n \geq 2,\ \aleph_n \text{ has the Super Tree Property })$.

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If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega} \text{ supercompact cardinals})$, then $Cons(ZFC + \aleph_{\omega+1} \text{ has the Strong Tree Property })$.

The Super Tree Property at Small Cardinals

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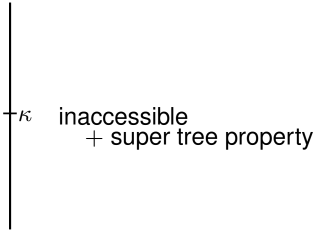
The Super Tree Property at Small Cardinals

Weiss

Let $n \geq 2$, if *Cons*($ZFC + \exists\kappa$ supercompact), then *Cons*($ZFC + \aleph_n$ has the Super Tree Property).



use $\mathbb{M}(\aleph_n, \kappa)$.

The Super Tree Property at Small Cardinals

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Let $n \geq 2$, if $Cons(ZFC + \exists\kappa \text{ supercompact })$, then $Cons(ZFC + \aleph_n \text{ has the Super Tree Property })$.

$\kappa = \aleph_{n+2}$
$+$ super tree property

use $\mathbb{M}(\aleph_n, \kappa)$.

The Super Tree Property at Small Cardinals

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If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then $Cons(ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property).

λ inaccessible
$+$ super tree property

κ inaccessible
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$\lambda = \aleph_3$
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$\kappa = \aleph_2$
$+$ super tree property ?

$$\mathbb{M}(\aleph_0, \kappa) * \mathbb{M}(\aleph_1, \lambda)$$

Cummings and Foreman's Iteration

$\langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals

At stage $n+1$, we force with $\mathbb{Q}_n$.

1. $\mathbb{Q}_n$ makes $\kappa_n = \aleph_{n+2}$ while preserving the super tree property at κ_n;
2. $\mathbb{Q}_n$ anticipates the tail of the iteration $Tail_{n+1}$ (using the Laver function L_n).

κ_n inaccessible
$+$ super tree property

κ_1 inaccessible
$+$ super tree property

κ_0 inaccessible
$+$ super tree property

κ_n inaccessible
$+$ super tree property

$\mathbb{Q}_0$

κ_1 inaccessible
$+$ super tree property

κ_0 $= \aleph_2$
$+$ super tree property

κ_n inaccessible
$+$ super tree property

$\mathbb{Q}_0 \ * \ \dot{\mathbb{Q}}_1$

κ_1 $= \aleph_3$
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$\kappa_n = \aleph_{n+2}$
$+$ super tree property

$\kappa_1 = \aleph_3$
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$$\mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * \cdots * \dot{\mathbb{Q}}_n * \cdots$$

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If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then
$Cons(ZFC + \forall n \geq 2,\ \aleph_n$ has the Super Tree Property).

Proof.

We check $V[G_\omega] \models \aleph_2$ has the super tree property. In $V[G_\omega]$, fix F an $(\aleph_2, \mu)$-tree and $D := \langle d_X;\ X \in [\mu]^{<\aleph_2}\rangle$ an F- level sequence. In that model $\kappa_0 = \aleph_2$. Fix an elementary embedding $j : V \to N$ such that:

- $cr(j) = \kappa_0$, $j(\kappa_0) > \sigma$ and ${}^\sigma N \subseteq N$, for σ large enough;
- $j(L_0)(\kappa_0)$ is a $\mathbb{Q}_0$-name for $Tail_1$

Step 1 : lift j to an elementary embedding $j : V[G_\omega] \to N[H^*]$
(hint: use $j(\mathbb{Q}_0) \upharpoonright \kappa_0 + 1 = \mathbb{Q}_0 * j(L_0)(\kappa_0)$);

Step 2 : find an ineffable branch b for D in $N[H^*]$

Step 3 : prove that $b \in V[G_\omega]$; (use two preservation lemmas). □

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The Strong Tree Property at $\aleph_{\omega+1}$

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The Strong Tree Property at $\aleph_{\omega+1}$

Magidor & Shelah 1996

If ν is a singular limit of strongly compact cardinals, then ν^+ satisfies the Tree Property.

Fontanella 2012 - Key Lemma

If ν is a singular limit of strongly compact cardinals, then ν^+ satisfies the Strong Tree Property.

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If $Cons(ZFC + \exists\langle\kappa_n\rangle_{n<\omega}$ supercompact cardinals), then $Cons(ZFC + \aleph_{\omega+1}$ has the Strong Tree Property).

Start with $\langle\kappa_n\rangle_{n<\omega}$ indestructible supercompact, let ν be the limit. Force to make κ_n the n-th successor of κ_0, let W be the resulting model.

We prove that in W there is $\mu < \kappa_0$ of cof. ω such that

$$\mathbb{L}(\mu) := Coll(\omega, \mu) \times Coll(\mu^+, < \kappa_0)$$

forces the strong tree property at ν^+ (i.e. $\aleph_{\omega+1}$).

By contrad. there is no such μ. For every μ..., let $\dot{F}_\mu$ be a counterexample, i.e. an $\mathbb{L}(\mu)$-name for a (ν^+, λ_μ)-tree with no cofinal branches. W.l.o.g $\lambda = \lambda_\mu$ for every μ.

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$$I := \{(a, b, \mu);\ \mu < \kappa_0 \ldots \text{ and } (a, b) \in \mathbb{L}(\mu)\}$$

For every $i = (a, b, \mu)$ in I, for every $X, Y \in [\lambda]^{<\nu^+}$ and $\zeta, \eta < \nu$ define

$$(X, \zeta)\ S_i\ (Y, \eta) \iff (a, b) \Vdash_{\mathbb{L}(\mu)} \dot{f}^X_\zeta <_{\dot{F}_\mu} \dot{f}^Y_\eta.$$

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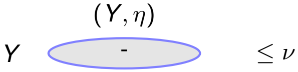

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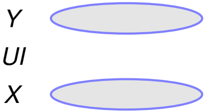

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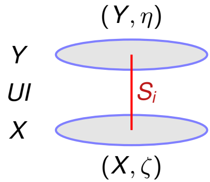

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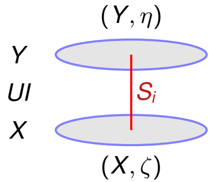

We want to prove that there is $i = (a, b, \mu) \in I$ and $\zeta < \nu$ such that $(X, \zeta)\ S_i\ (Y, \zeta)$ for cofinally many X, Y with $X \subseteq Y$. In other words we are looking for a

cofinal S_i-branch for S.

Step 1: (use κ_0 superc. in W) There is $n < \omega$ and D cofinal such that $R := \{ S_i \restriction D \times \kappa_n \}_i$ is still a system.

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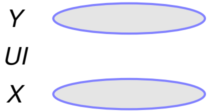

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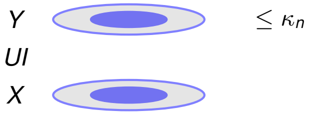

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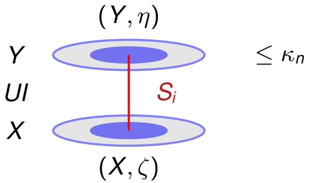

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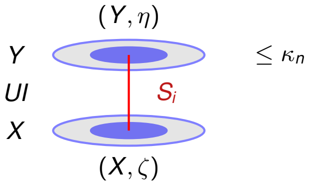

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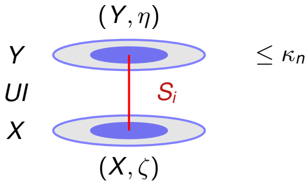

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Step 3: A cofinal branch S_i-branch for R (hence for S) already existed in W (use a third Preservation Lemma).

Future work

- What cardinals can satisfy those properties?
- How can we use them?
- Can we find similar characterizations of other large cardinals?

Thank you.